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(University of Choice)

MASINDE MULIRO UNIVERSITY OF SCIENCE AND TECHNOLOGY (MMUST)

UNIVERSITY EXAMINATIONS
2021/2022 ACADEMIC YEAR

THIRD YEAR SECOND SEMESTER MAIN EXAMINATIONS

FOR THE DEGREE OF BACHELOR OF SCIENCS

COURSE CODE: SPH 315

COURSE TITLE: MATHEMATICAL METHODS FOR PHYSICS

DATE: TUESDAY 26TH APRIL, 2022 **TIME:** 8:00 AM - 10:00 AM

INSTRUCTIONS TO CANDIDATES

TIME: 2 Hours

Answer question ONE and any TWO of the remaining.
Symbols used bear the usual meaning.

MMUST observes ZERO tolerance to examination cheating

QUESTION ONE (30 MARKS)

(a) Show that $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10} = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$ (5 marks)

(b) Evaluate the contour integral $\int_C f(z)dz$ using the parameter representation for C, where $f(z) = z^2 - 1/z$ and the curve C is (5 marks)

(i) the semicircle $z = 2e^{i\theta}$ $\quad 0 \le \theta \le \pi$

(ii) the semicircle $z = 2e^{i\theta}$ $\quad \pi \le \theta \le 2\pi$

(c) Verify that the associated Legendre function $P_{2,0}(x) = \frac{1}{2}(3x^2 - 1)$ is a solution of the associated Legendre's equation (5 marks)

$$(1-x^2)\frac{d^2\Theta(x)}{dx^2} - 2x\frac{d\Theta(x)}{dx} + l(l+1)\Theta(x) - \frac{m^2}{(1-x^2)}\Theta(x) = 0; \qquad \Theta(x) \to P_{l,m}(x)$$

(d) Gamma functions $\Gamma(p)$ is defined by $\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}dx$, show that (5 marks)

(i) $\Gamma(p+1) = p\Gamma(p)$

(ii) $\Gamma(1) = 1$

(e) Compute the following limits (5 marks)

$$\lim_{z \to -i} \frac{iz^3 + 1}{z^2 + 1}$$

(f) Deduce that the free space Green function for the divergence of the electric field in the presence of a charge distribution can be expressed by $\nabla \cdot \mathbf{E} = \frac{\rho(r)}{\varepsilon_0}$ (Gauss's Law)

(5 marks)

QUESTION TWO (20 MARKS)

(a) (i) Deduce the Cauchy-Riemann condition for the analyticity of a complex function $f(z) = u(x,y) + iv(x,y)$. (8 marks)

(ii) Verify that the function ($u(x,y) = \ln(x^2 + y^2)$ is harmonic and calculate a conjugate harmonic function v (7 marks)

$\int 2x/(x^2+y^2)dx = 2\arctan y/x$

(b) Apply De Moivre's theorem to express $\sin(3\theta)$ and $\cos(3\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$

(5 marks)

QUESTION THREE (20 MARKS)

(a) The Taylor series expansion for a complex function $f(z)$ with center at a is given by

$$f(z) = f(a) + \frac{(z-a)}{1!}f'(a) + \frac{(z-a)^2}{2!}f''(a)\cdots\cdots\cdots\frac{(z-a)^n}{n!}f^n(a)$$

Find the Taylor series expansion for

(i) $f(z) = \sin z$ (4marks)

(ii) $f(z) = \cos z$ (4 marks)

(b) The general solution of the Bessel equation $z^2y'' + zy' + \left(z^2 - \frac{1}{2}\right)y = 0$ with $v = \frac{1}{2}$ is $y(z) = c_1 J_{\frac{1}{2}}(z) + c_2 J_{-\frac{1}{2}}(z)$, where

$$J_{\pm\frac{1}{2}}(z) = z^{\pm\frac{1}{2}}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{2^{2n\pm\frac{1}{2}}\, n!\,\Gamma\left(1+n\pm\frac{1}{2}\right)}$$

Show that

$$y(z) = c_1 J_{\frac{1}{2}}(z) + c_2 J_{-\frac{1}{2}}(z) = c_1\sqrt{\frac{2}{\pi z}}\sin z + c_2\sqrt{\frac{2}{\pi z}}\cos z$$

(12 marks)

QUESTION FOUR (20 MARKS)

(a) Find the appropriate Green function $G(x,z) = \sum_{n=0}^{\infty}\frac{1}{\lambda}y_n(x)y_n^*(z)$ for the equation $y'' + \frac{1}{4}y = f(x)$ with the boundary condition $y = 0$ and $y = \pi$. (10 marks)

(b) Hermite polynomial $H_n(x)$ are given by the Rodrigues formula $H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{x^2}\right)$

(i) Calculate the Hermite polynomials $H_2(x)$ and $H_3(x)$ (6 marks)

(ii) Prove the recurrence relation $2xH_n(x) - H_n'(x) = H_{n+1}(x)$ (4 marks)

QUESTION FIVE (20 MARKS)

(a) An important application of the Gama function is in the evaluation of definite integrals where

$$\Gamma(p) = 2\int_0^{\infty} x^{2(p-1)}e^{-x^2}dx$$

Evaluate the normal Gaussian distribution of a statistical measurement of a quantity X, centered at a mean X_0, having a random rms error spread σ (8 marks)

$$y(x) = Ne^{-x^2/2};\quad where\quad x = {(N - N_0)}/{\sigma}$$

(b) (i) Laguerre polynomials $L_j(x)$ and associated Laguerre polynomials $L_j^k(x)$ of any order can be calculated using the generating functions

$$L_j(x) = e^x \frac{d^j}{dx^j} e^{-x} x^j \quad ; \qquad L_j^k(x) = (-1)^k \frac{d^k}{dx^k} L_{j+k}(x)$$

Calculate the associated Laguerre polynomials $L_1^1(x)$ and $L_2^1(x)$ (8 marks)

(ii) Evaluate the radial wave functions $R_{1,0}(r)$ and $R_{2,0}(r)$ using the normalized equation (4 marks)

$$R_{n,l}(r) = \sqrt{\left(\frac{2Z}{na_0}\right)^3 \frac{(n-l-1)!}{2n((n+l)!)^3}} e^{-Zr/na_0} \left(\frac{2Zr}{na_0}\right)^l L_{n+l}^{2l+1}\left(\frac{2Zr}{na_0}\right)$$