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(The University of Choice)

MASINDE MULIRO UNIVERSITY OF SCIENCE AND TECHNOLOGY

UNIVERSITY EXAMINATIONS

2021/2022 ACADEMIC YEAR

FOURTH YEAR SECOND SEMESTER MAIN EXAMINATIONS

FOR THE DEGREE

IN

BACHELOR OF SCIENCE (SMT, SME)

COURSE CODE: **MAT 402**

COURSE TITLE: **MEASURE THEORY**

DATE: 26/04/2022 **TIME: 12.00 NOON – 2.00PM**

INSTRUCTIONS TO CANDIDATES

- Section A is compulsory any other THREE questions from section B
- Do all the rough work in the answer booklet

TIME: 2 hours

QUESTION ONE (30 MARKS)

a) Let a sequence $\{f_n\}, n \in \mathbb{N}$ of measurable functions be dominated by an integrable function g, that is $|f_n(x)| \leq g(x)$, holds for every $n \in \mathbb{N}$ and every $x \in E$ and let $\{f_n\}$ converges pointwise to a function f, that is $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in E$. Show that $\int_E f = \lim_{n\to\infty} \int_E f_n$. (6 Marks)

b) If E_1 and E_2 are measurable sets, show that (5 Marks)

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$$

c) Define the following terms:

i) Measurable function

ii) Lebesgue outer measure (4 Marks)

d) Let $\{A_n\}$ be a countable collection of sets of real numbers. Prove that (5 Marks)

$$m^*(\cup A_n) \leq \sum m^*(A_n)$$

e) Let E be a set of rationals in $[0,1]$. Show that the characteristic function

$$\chi_{E(X)} = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

is measurable. (4 Marks)

f) Evaluate the Lebesgue integral of the function $f: [0,1] \to \mathbb{R}$. (6 Marks)

$$f(x) = \begin{cases} \dfrac{1}{x^{1/3}} & if\ 0 \leq x \leq 1 \\ 0 & if\ x = 0 \end{cases}$$

QUESTION TWO (20 MARKS)

a) If th sequence $\{f_n\}$ converges in measure to the function f, show that the limit function f is unique a.e. (5 Marks)

b) Let E be a measurable set wth finite measure and let $\{f_n\}$ be a sequence of measurable functions converging almost everywhere to a real valued function f defined on a set E. Prove that given $\varepsilon > 0$ and $\delta > 0$, there corresponds a measurable subset A of E with $m(A) < \delta$ and an integer N such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in E - A$ and $n \geq N$.

(6 Marks)

c) Let f and g be any two functions which are equal almost everywhere in E. Show that if f is measurable so is g. (6 Marks)

d) Let $\{f_n\}$ be a sequence of measurable functions such that $\lim_{n\to\infty} f_n = f$ a.e. Then f is a mesurable function. (3 Marks)

QUESTION THREE (20 MARKS)

a) Show that if $A \subseteq B$, then $m^*(A) \leq m^*(B)$. (4 Marks)

b) Prove that every Borel set is measurable. (5 Marks)

c) Let $\{E_n\}$ be an increasing sequence of measurable sets, that is, a sequence with $\subset E_{n+1}$ for each n. Let $m(E_1)$ be finite. Prove that $m(\bigcup_{i=1}^{\infty} E_i) = \lim_{n\to\infty} m(E_i)$. (7 Marks)

d) Show that if $m(E_1 \Delta E_2) = 0$ and E_1 is measurable, then E_2 is measurable. Moreover $m(E_2) = m(E_1)$. (4 Marks)

QUESTION FOUR (20 MARKS)

a) Let E be a set with $m^*(E) < \infty$. Show that E is measurable if and only if given $\varepsilon > 0$, there is a finite union B of open intervals such that $m^*(E \Delta B) < \varepsilon$. (10 Marks)

b) If $\{f_n\}$ is a sequence of measurable functios converging to f. Show that f is also measurable. (4 Marks)

c) Let f be a measurable function defined on where E_1 and E_2 are measurable on $E_1 \cup E_2$ if and only if $f|_{E_1}$ and $f|_{E_2}$ are measurable. (6 Marks)

QUESTION FIVE (20 MARKS)

a) Show that if $E_1, E_2, \dots, E_n$ are disjoint measurable subsets of E then every linear combination $\phi = \sum_{i=1}^{n} c_1 \chi_{E_i}$ with real coefficients $c_1, c_2, \dots, c_n$ is a simple function and $\int \phi = \sum_{i=1}^{n} c_i m(E_i)$. (6 Marks)

b) Let $\{f_n\}$ be an increasing sequence on non-neagative measurable functions on E. If $\{f_n\} \to f$ point-wise a.e. on E, show that $\lim_{n\to\infty} \int_E f_n = \int_E f$. (5 Marks)

c) If $\{f_n\}$ is a sequence of non-negative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a set E, then $\int_E f \leq \lim\_\_\_ \int_E f_n$. (5 Marks)

d) If f and g are non-negative measurable functions, show that $\int_E (f+g) = \int_E f + \int_E g$.

(4 Marks)