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(University of Choice)

MASINDE MULIRO UNIVERSITY OF SCIENCE AND TECHNOLOGY (MMUST)

UNIVERSITY REGULAR EXAMINATIONS
2022/2023 ACADEMIC YEAR

FIRST YEAR SECOND SEMESTER (MAIN CAMPUS)

FOR THE DEGREE

OF

MASTER OF SCIENCE (PURE MATHEMATICS)

COURSE CODE: MAT 813E

COURSE TITLE: OPERATOR THEORY I

DATE: Wednesday 26th April 2023

TIME: 2.00 pm – 5.00 pm

INSTRUCTIONS: Answer any THREE questions

QUESTION ONE (20 MARKS)

a) Let X be a finite dimensional normed linear space and T be a bounded linear operator on X. Show that the spectrum of T consists only of eigenvalues of T i.e. $\sigma(T) = P\sigma(T)$.

(4 marks)

b) Let X be a Banach space and T be a bounded linear operator in X. Show that the resolvent function $R(\,;T): \rho(T) \to B(X)$ is analytic and that the resolvent set $\rho(T)$ is open in $\mathbb{C}$.

(6 marks)

c) Let X be a Banach space and p be a polynomial with complex coefficients. Let $T \in B(X)$. Show that $\sigma(p(T)) = p(\sigma(T))$ where $p(\sigma(T)) = \{p(\lambda) : \lambda \in \sigma(T)\}$.

(6 marks)

d) Let X be a complex Banach space and $T \in B(X)$ which is invertible. Show that $\sigma(T^{-1}) = \{\sigma(T)\}^{-1}$ where $\{\sigma(T)\}^{-1} = \{\lambda^{-1} : \lambda \in \sigma(T)\}$.

(4 marks)

QUESTION TWO (20 MARKS)

a) Let H be a complex Hilbert space and $T \in B(H)$. Show that $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ exists and equals the spectral radius of T i.e. $r(T)$.

(9 marks)

b) Let H be a Hilbert space and $T \in B(H)$. Show that the following statements are equivalent.

i. There is a $\lambda \in \Pi(T)$ such that $|\lambda| = \|T\|$.

ii. $\|T\| = Sup\{|\langle Tx, x\rangle| : x \in H \text{ and } \|x\| = 1\}$. (7 marks)

c) Let H be Hilbert space and $T \in B(H)$ be normal. Show that the approximate point spectrum of T equals the spectrum of T i.e. $\Pi(T) = \sigma(T)$.

(4 marks)

QUESTION THREE (20 MARKS)

a) Let H be a Hilbert space and $T \in B(H)$ be normal. Show that the spectral radius of T equals the norm of T i.e. $r(T) = \|T\|$.

(6marks)

b) Let X be a complex Banach space and $T \in B(X)$. Show that there is a $\lambda \in \sigma(T)$ such that $|\lambda| = r(T)$. (Note that when X is a complex Banach space, then $\sigma(T) \neq \phi$).

(4 marks)

c) Let H be a Hilbert space and $T \in B(H)$ be normal. Show that the residual spectrum of T is void i.e. $R\sigma(T) = \phi$. (4 marks)

d) Let H be a Hilbert space and $T \in B(H)$. Show that a closed linear subspace M of H is T-invariant if and only if $M^{\perp}$ is T^*-invariant.

(3 marks)

e) Let H be a Hilbert space and M be a closed linear subspace of H. Let $T \in B(H)$. Show that M is T-invariant if and only if $PTP = TP$ where P is orthogonal projector on H onto M.

(3 marks)

QUESTION FOUR (20 MARKS)

a) Let H be a Hilbert space and $T \in B(H)$. Let M be a closed linear subspace of H and P is orthogonal projector on H onto M. Show that the following statements are equivalent

i. M reduces T.

ii. M reduces T^*.

iii. $P \leftrightarrow T$.

iv. $P \leftrightarrow T^*$.

v. $M^{\perp}$ reduces T.

vi. $M^{\perp}$ reduces T^*. (7 marks)

b) Let H be a Hilbert space and $T \in B(H)$ be positive. Show that T is self-adjoint but the converse need not be true.

(5 marks)

c) Let H be a Hilbert space and P, Q be orthoprojectors on H. Show that PQ is an orthoprojector if and only if $P \leftrightarrow Q$, also show that in this case $\Re_{PQ} = M \cap N$ where M and N are the closed linear subspaces of H onto which P and Q project.

(5 marks)

d) Let $T, S \in B(X)$ and $\{T_\alpha : \alpha \in \Lambda\}$ be a summable family of elements of $B(X)$ such that $\sum_{\alpha \in \Lambda} T_\alpha = T$. Show that $\{ST_\alpha : \alpha \in \Lambda\}$ and $\{T_\alpha S : \alpha \in \Lambda\}$ are summable to ST and TS respectively. (3 marks)

QUESTION FIVE (20 MARKS)

a) Let H be a Hilbert space and $T \in B(H)$ be positive. Show that

i. $|\langle Tx, y\rangle|^2 \le \langle Tx, x\rangle\langle Ty, y\rangle$ for all $x, y \in H$.

ii. $\|Tx\|^2 \le \|T\|\langle Tx, x\rangle$ for all $x \in H$. (5 marks)

b) If $S \ge 0, T \ge 0$ and $S \leftrightarrow T$, show that

i. $ST \ge 0$

ii. $\sqrt{ST} = \sqrt{S}\sqrt{T}$. (4 marks)

c) Let H, K be Hilbert spaces. A linear operator $U \in B(H, K)$ is a partial isometry if and only if it is isometric on the orthogonal complement of its null space η_U.

(5 marks)

d) Let H be a Hilbert space and $T \in B(H)$ be normal. Show that T has a polar decomposition $T = US$ where U is unitary and $S \geq 0$. Moreover show that $U \leftrightarrow S$ (i.e. $US = SU = T$). If T is 0-invertible, show that the decomposition is unique.

(6 marks)

--------GOOD LUCK--------

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