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(University of Choice)

MASINDE MULIRO UNIVERSITY OF SCIENCE AND TECHNOLOGY (MMUST)

UNIVERSITY EXAMINATIONS

2022/2023 ACADEMIC YEAR

FIRST YEAR SECOND SEMESTER EXAMINATIONS

FOR THE DEGREE
OF

MASTER OF SCIENCE (PURE MATHEMATICS)

COURSE CODE: MAT 808
COURSE TITLE: FUNCTIONAL ANALYSIS II

DATE: Thursday 20th April 2023 **TIME:** 8.00 am – 11.00 am

INSTRUCTIONS TO CANDIDATES
Answer ANY THREE questions

Time: 3 hours

This Paper Consists of 3 Printed Pages. Please Turn Over.

QUESTION ONE (20 MARKS)

Let H be a Hilbert space and $T \in B(H)$ be a bounded linear operator on H. Show that

i. The adjoint of T, T^* is unique, linear and bounded (10 marks)
ii. $\| T^*\| = \|T\|$ (3 marks)
iii. $\| T^*T\| = \|T\|^2$ (3 marks)
iv. $(T^*)^* = T$ (1 mark)
v. $(ST)^* = T^*S^*$ (3 marks)

QUESTION TWO (20 MARKS)

a. Let H be a Hilbert space and let $T \in B(H)$ be a bounded linear operator on H. Show that the resolvent set $\rho(T)$ for T is an open set in $\mathbb{C}$ and that the spectrum $\sigma(T)$ for T is a closed set $\mathbb{C}$. (5 marks)
b. Let (e_n) be an orthonormal basis for the infinite dimensional separable Hilbert space H. Define T, a one sided operator on H with respect to this basis and determine its spectrum.
c. Let T be a compact, self-adjoint linear operator on a finite dimensional Hilbert space H. Show that H admits an orthonormal basis (e_n) consisting of eigenvectors for T. Discuss the finite dimensional and infinite dimensional cases. (Spectral Theorem for Compact Operators)

QUESTION THREE (20 MARKS)

a. Let P be a bounded self-adjoint linear operator on the Hilbert space H and assume that P is idempotent. Show that P is the orthogonal projection into the closed linear subspace $M = \{z \in H \mid Pz + z\}$ (fixed space for P) (10 marks)
b. Let L be a closed subspace of a Hilbert Space H. Show that H can be expressed as a direct sum of L and its complement, $H = L \oplus L^{\perp}$. (10 marks)

QUESTION FOUR (20 MARKS)

a. State and prove the Parallelogram Law. (3 marks)
b. Prove that the norm $\|\cdot\|$, on a linear space E induced by an inner product $\langle , \rangle$ in it if and only if the norm satisfies the Parallelogram Law and that in this case the inner product is given by the Polarization Identity. (17 marks)

QUESTION FIVE **(20 MARKS)**

a. Show that if a sequence of bounded linear operators (A_n) is a Cauchy sequence at every point of a Banach space, then the sequence $\{\|A_n\|\}$ of these operators is uniformly bounded (Banach-Steinhaus Theorem).

b. Let X and Y be two Banach spaces. Let $\{A_n\}$ be a sequence of bounded linear operators mapping $X \rightarrow Y$ such that $\{A_n x\}$ is a bounded subset for each $x \in X$. Show that the sequence $\{\|A_n x\|\}$ of norms of these operators is bounded (Uniform Boundedness Principle).

c. Let $X = \{x = \{x_j\}$: only a finite number of x are non – zero$\}$ and $\|x\| = \sup_j |x_j|$. Consider the mapping which is a linear functional defined by $f_n(x) = \sum_{j=1}^{n} x_j$. Determine whether or not the sequence of functionals satisfies the Uniform Boundedness Principle.