*(University of Choice)*

# MASINDE MULIRO UNIVERSITY OF SCIENCE AND TECHNOLOGY (MMUST)

## UNIVERSITY REGULAR EXAMINATIONS
## 2022/2023 ACADEMIC YEAR

### FIRST YEAR FIRST SEMESTER (MAIN CAMPUS)

### FOR THE DEGREE

### OF

### DOCTOR OF PHILOSOPHY (PURE MATHEMATICS)

COURSE CODE:       MAT 901

COURSE TITLE:      TOPICS IN OPERATOR THEORY I

DATE:              Wednesday 26th April 2023

TIME:              8.00 a.m – 11.00 a.m

INSTRUCTIONS:      Answer any THREE questions

## QUESTION ONE (20 MARKS)

a) Let X be a Banach space and $T \in B(X)$. Define the operator $e^T$ and show that if $S \in B(X)$ and S commutes with T, then $e^{T+S} = e^T e^S$ and hence show that $e^T$ is invertible and that its inverse is $e^{-T}$. (11 marks)

b) If H is a Hilbert space and $T \in B(H)$ is normal and if TS=ST for some $S \in B(H)$, show that $T^*S = ST^*$. (5 marks)

c) Let H be a Hilbert space and $T \in B(H)$. Show that the boundary of the spectrum of T is contained in the approximate point spectrum of T. (4 marks)

## QUESTION TWO (20 MARKS)

a) Let P, Q be diagonal operators with diagonals $\{\alpha_n\}, \{\beta_n\}$ such that $|\alpha_n| = |\beta_n| \, \forall \, n$. Show that the weighted shifts A=SP, B=SQ (where S is a shift) are unitarily equivalent i.e. there exists a unitary operator D on H such that DA=BD. (8 marks)

b) Let A, B be unilateral weighted shifts with non-zero weights $\{\alpha_n\}, \{\beta_n\}$. Show that a necessary and sufficient condition for A to be similar to B is that the sequence of quotients $\left( \left| \dfrac{\alpha_0 \text{-----} \alpha_n}{\beta_0 \text{------} \beta_n} \right| \right)_{n=1}^{\infty}$ be bounded away from both 0 and $\infty$ (12 marks)

## QUESTION THREE (20 MARKS)

a) Given a unilateral weighted shift A with non-zero weights $\{\alpha_n\}$, compute norm of A and its spectral radius r(A). (7marks)

b) What is the spectrum of the unilateral shift and what are its parts (point spectrum, compression spectrum and the approximate point spectrum)? Answer the same for the adjoint of the unilateral shift. (13 marks)

## QUESTION FOUR (20 MARKS)

a) For which sequences $(P_n)$ of positive numbers will there exist an operator A such that $\|A^n\| = P_n$ for all $n \in \mathbb{N}$. (7 marks)

b) Construct a quasinilpotent operator for which the point spectrum is void. (7marks)

c) Let A be a unilateral weighted shift with positive weights $\{\alpha_n\}$, where $\alpha_n \to 0$ as $n \to \infty$. Show that $\sigma(A) = \{0\}$ and $P\sigma(A) = \phi$ (6marks)

## QUESTION FIVE (20 MARKS)

a) Consider $L^2(\mu)$ for a $\mu$ which is $\sigma$-finite. If a bounded operator A on $L^2(\mu)$ is induced by a multiplier $\phi : X \to \mathbb{C}$, show that $\phi$ is measurable and essentially bounded.

(7 marks)

b) Show that $\sigma(A) = \Sigma$, where A is the multiplication operator and $\Sigma$ is the essential range of the multiplier

(8 marks)

c) Show that the set of invertible operators in B(H) is open and that the map $T \mapsto T^{-1}$ is continuous in B(H) in the norm topology of B(H).

(5 marks)

--------GOOD LUCK--------

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