(University of Choice)

MASINDE MULIRO UNIVERSITY OF SCIENCE AND TECHNOLOGY

UNIVERSITY EXAMINATIONS
2021/2022 ACADEMIC YEAR
(MAIN CAMPUS)

FIRST YEAR SECOND SEMESTER REGULAR EXAMINATIONS FOR THE DEGREE OF MASTER OF SCIENCE (STATISTICS)

COURSE CODE: **STA 803**

COURSE TITLE: STATISTICAL INFERENCE I

DATE: Monday 25th April, 2022 **TIME:** 09:00 - 12:00 Hrs

INSTRUCTIONS TO CANDIDATES

1. *This paper consists of* ***FIVE QUESTIONS.***
2. ***ANSWER ANY THREE QUESTIONS***
3. *In each question, show your* **working clearly**
4. *There will be marks for proper working even if the answer is wrong*
5. *Calculators and Statistical tables may be used*

Time: 3 hours

MMUST observes ZERO tolerance to examination cheating

This Paper Consists of 3 Printed Pages. Please Turn Over.

Question 1

a. Let $\hat{\theta}_n$ be an estimator of θ. Show that, the mean squared error (MSE) of $\hat{\theta}$ can be written as $MSE\left(\hat{\theta}_n\right) = Bias\left(\hat{\theta}_n\right)^2 + Var\left(\hat{\theta}_n\right)$ (3 marks)

b. Maximum likelihood estimators have a desirable *invariance* property. That is, if $\hat{\theta}_n$ is the MLE of θ, and $\eta = g(\theta)$ is some transformation, then the MLE of η is $\hat{\eta} = g\left(\hat{\theta}_n\right)$. Explain the intuition behind this fact. (4 marks)

c. If $X = X_1, X_2, \ldots, X_n$ are i.i.d. Bernoulli(θ).

   i. Show that $X \sim Ber(\theta)$ belongs to an exponential family with $\eta(\theta) = \log\frac{\theta}{1-\theta}$, hence find the sufficient statistic for θ. (4 marks)

   ii. Find the canonical link and write down the formula for θ_i in terms of a predictor variable x_i and a parameter β. (4 marks)

d. Let $X_1, X_2, \ldots, X_n$ be iid from the exponential distribution with density

$$f(x) = \begin{cases} \frac{1}{\theta}e^{-\frac{x}{\theta}}; \ x > 0 \ and \ \theta > 0 \\ 0; \ o.w \end{cases}$$

Find the approximations to $E\left[\sqrt{\overline{x}}\right]$ and $Var\left[\sqrt{\overline{x}}\right]$. (5 marks)

Question 2 **(20 marks)**

a. Let $\underline{X} = X_1, X_2, \ldots, X_n$ are iid $U(0,\theta)$. Show that;

   i. the MLE of θ is $\hat{\theta} = \max_{1 \le i \le n} X_i$ (3 marks)

   ii. $n\left(\theta - \hat{\theta}\right)$ converges in distribution to $Exp(\theta)$ (4 marks)

b. Let $X_1, X_2, \ldots, X_n$ be iid as $N(0,1)$. Consider the two estimators

$$T_n = \begin{cases} \overline{X}_n; \ if \ S_n \le a_n \\ n; \ if \ S_n > a_n \end{cases}$$

where $S_n = \sum (x_i - \overline{x})^2$, $P(S_n > a_n) = \frac{1}{n}$, $T_n' = \frac{(X_1, X_2, \ldots, X_{k_n})}{k_n}$, with k_n the largest integer $\le \sqrt{n}$

Show that the asymptotic efficiency of T_n' relative to T_n is zero. (5 marks)

c. Suppose $X = X_1, X_2, \ldots, X_n$ are iid $N\left(\theta, \theta^2\right)$.

   i. Show that $N\left(\theta, \theta^2\right)$ has an exponential family form. (3 marks)

   ii. Find the minimal sufficient statistic for θ. (2 marks)

   iii. Show that your minimal sufficient statistic is not complete (3 marks)

Question 3 **(20 marks)**

a)

   i. What is the effect of increasing the level of confidence on the constructed width of the confidence interval for population mean μ? (2 marks)

   ii. You are designing an experiment that involves taking a random sample from an infinitely large, normally distributed population with known standard deviation $\sigma = 1.5$. Your experiment requires a confidence-interval estimate that will give 95% confidence that the margin of error of the estimate of μ is $E = 0.92$. How large should the sample be? (3 marks)

b) Suppose $X_1, X_2, ..., X_n$ are iid $Poisson(\lambda)$.

i. Show that $\overline{x}$ is the UMVU estimator for λ. (6 marks)

ii. For $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2$, show that $E[S^2] = E[\overline{x}] = \lambda$. (3 marks)

c) The following data give the distribution of life (t) in hours of 200 electric bulbs of a certain manufacturing company. Assuming that the distribution of t follows the exponential law

$$f(t,\theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{t}{\theta}}; \ t > 0 \ and \ \theta > 0 \\ 0; \ o.w \end{cases}$$

Find the value of the character θ by the method of (i) MME and (ii) MLE. How do the two estimators compare? (6 marks)

Question 4 **(20 marks)**

a) Suppose that $X_1, X_2, ..., X_n$ are i.i.d. with density $p_\theta(x) = g(x,\theta) = \frac{1}{\pi(1+x^2)}; x \in \Re$, where g is a Cauchy density. Assuming that g' exists and the other regularity conditions hold, find the CRLB for $\tau(\theta) = e^{-\theta}$. (6 marks)

b) Let $X_1, X_2, ..., X_n$ are i.i.d. with density $Weibull(\alpha, \beta)$, such that,

$$f(x,\alpha,\beta) = \begin{cases} \alpha\beta(\alpha x)^{\alpha-1} e^{-(\beta x)^\alpha} & ; x \geq 0 \\ 0 \ ; \ otherwise \end{cases}$$

If α is known, find the MLE of β. (5 marks)

c) Let $\underline{X} = X_1, X_2, ..., X_n$ be a random sample of size n from a population with normal distribution with mean μ and variance σ^2. Find the UMVUE of μ and σ^2. (8 marks)

Question 5 **(20 marks)**

Suppose $X_1, X_2, ..., X_n$ is a sample of n observations from i.i.d. random variables having a $Poisson(\mu)$ distribution.

a) We wish to make inferences on $g(\mu) = \log(\mu)$. Show that the MLE of:

i. $\hat{\mu} = \overline{x}$ (3 marks)

ii. $g(\mu) = \log(\mu)$ is $g(\hat{\mu}) = \log(\overline{x})$ (2 marks)

iii. Derive the asymptotic distribution for $g(\hat{\mu})$. (6 marks)

iv. Now suppose that the number of radioactive emissions per second from a source is thought to follow a $Poisson(\mu)$ distribution. Suppose 100 independent counts of the number of emissions give a mean count of $\overline{x} = 1.2$. Calculate a 99% confidence interval for $\log(\mu)$, and convert this back to a 99% confidence interval for μ. (4 marks)

b) Determine the UMVU estimator of $P[X_i = 0] = e^{-\lambda}$ and hence calculate the variance of its estimator up to terms of order $\frac{1}{n}$ (5 marks)