-160

(University of Choice)

# MASINDE MULIRO UNIVERSITY OF SCIENCE AND TECHNOLOGY

# (MMUST)

## UNIVERSITY EXAMINATIONS

## 2021/2022 ACADEMIC YEAR

## SECOND YEAR SECOND SEMESTER EXAMINATIONS

FOR THE DEGREE
OF

## BACHELOR OF SCIENCE (MATHEMATICS WITH IT)
## AND
## BACHELOR OF TECHNOLOGY (MATHEMATICS AND COMPUTER SCIENCE)

**COURSE CODE:** MAT 206
**COURSE TITLE:** ALGEBRAIC STRUCTURES

**DATE:** Tuesday, 26th April 2022 **TIME:** 12 Noon - 2.00 pm

INSTRUCTIONS TO CANDIDATES

Answer question ONE (COMPULSORY) and any other TWO questions

Time: 2 hours

## QUESTION ONE (COMPULSORY) – 30 MARKS

a. Let $(G, *)$ be a group. Prove that for all $a \in G$, there exists a unique inverse element to a. (5 marks)

b. Suppose a, b and c are integers and suppose $a \neq 0$ and $b \neq 0$, show that if a divides b and b divides c then a divides c. (4 marks)

c. Factorize 38808 into primes and state the prime order of each of the prime factors. (4 marks)

d. In $S_6$ with $\sigma = (135)(26)$ and $\tau = (13456)$, calculate $\sigma\tau$ and $\tau\sigma\tau$. (5 marks)

e. Write out the multiplication tables for addition and multiplication modulo 4. Explain why $(\mathbb{Z}_4, +, \cdot)$ is a ring. (5 marks)

f. Let $(G, *)$ be a group. Prove that for all $a, b \in G$, $(a * b)^{-1} = b^{-1} * a^{-1}$. (6 marks)

g. Define the following terms in the context of Algebraic Structures:
   - i. A composite number
   - ii. An abelian group
   - iii. A subgroup
   - iv. A field
   - v. Greatest common divisor (5 marks)

## QUESTION TWO – 20 MARKS

a. The table below represents a binary action on the set $\{a, b, c, d, e\}$. Use it to answer the questions that follow.

| * | a | b | c | d | e |
|---|---|---|---|---|---|
| a | a | b | c | b | d |
| b | b | c | a | e | c |
| c | c | a | b | b | a |
| d | b | e | b | e | d |
| e | d | b | a | d | c |

   - i. Compute $b * d$, $c * c$ and $[(a * c) * e] * a$. (4 marks)
   - ii. Compute $(a * b) * c$ and $a * (b * c)$. Based on this computation would you say that the binary operation $*$ is associative? Explain your answer. (4 marks)
   - iii. Compute $(b * d) * c$ and $b * (d * c)$. Based on this computation, explain whether or not $*$ is associative. (3 marks)
   - iv. Is $*$ commutative? Why? (2 marks)

b. Define a binary operation $*$ on $\mathbb{Q}$, the set of rational numbers by $a * b = ab + 1$. Determine whether or not the binary operation $*$ is
   - i. Commutative
   - ii. Associative. Explain your answer in each case. (4 marks)

c. Prove that the identity element in a group is unique. (3 marks)

## QUESTION THREE – 20 MARKS

a. Let S be the set of all real numbers except -1. Define $*$ on S by

$$a * b = a + b + ab$$

i. Show that $*$ is a binary operation on S. (1 mark)

ii. Show that $(S, *)$ is a group. (15 marks)

iii. Find the solution of the equation $2 * x * 3 = 7$. (4 marks)

## QUESTION FOUR – 20 MARKS

a. Prove that every cyclic group is abelian. (9 marks)

b. Let $M_2(\mathbb{R})$ be the set of two by two square matrices whose entries are real numbers. Determine all the elements of the cyclic subgroup of $M_2(\mathbb{R})$ generated by $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. (3 marks)

c. Prove the left cancellation property in a group that if $au = av$ for all $a$, $u$ and $v$ in G, then $u = v$ (3 marks)

d. Prove, by the method of mathematical induction that, $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$, for all integers $n \geq 1$. (5 marks)

## QUESTION FIVE – 20 MARKS

a. Determine the greatest common divisor of 22, 471 and 3,266 and express that gcd as a linear combination of the two numbers. (7 marks)

b. Prove that a subgroup of a cyclic group is also cyclic. (13 marks)