10

(The University Of Choice)

MASINDE MULIRO UNIVERSITY OF SCIENCE AND TECHNOLOGY (MMUST)

(MAIN EXAMINATIONS)

UNIVERSITY EXAMINATIONS
2021/2022 ACADEMIC YEAR

FIRST YEAR SECOND SEMESTER EXAMINATIONS

FOR THE DEGREE
OF
MASTER OF SCIENCE (PURE MATHEMATICS)

COURSE CODE: MAT 827

COURSE TITLE: COMMUTATIVE ALGEBRA

DATE: TUESDAY 26TH APRIL, 2022 TIME: 8.00 A.M -11.00 A.M

Instructions to candidates:
Answer any Three Questions

Time: 3 hours

QUESTION ONE (20 MARKS)

a) Let R be a unital non-zero ring, which is a field. Suppose $f: R \to S$ where , $S \neq \{0\}$ is a homomorphism, show that f is $1-1$. **[4 Marks]**

b) Define Nilpotent Elements and show that the set N of Nilpotent elements of a ring R forms an ideal of R. **[5 Marks]**

c) Describe the structure of the ring $\mathbb{Z}_4$ completely showing that it is primary, completely primary and Galois. **[5 Marks]**

d) Let R be a valuation ring. Then show that the following statements are true:
   i. R is Local
   ii. R is integrally closed
   iii. If H is a quotient field of R such that $H \subseteq K$, then R is a valuation with respect to H. **[6 Marks]**

QUESTION TWO (20 MARKS)

a) What is a Local Ring? Give Three examples of local rings. **[5 Marks]**

b) Let R be a ring with unity and $I_1, I_2, \ldots, I_n$ be a sequence of ideals of R. Let $f: R \to R/_{\mathbb{Z}} \prod_{i=1}^{n} I_i$ be a map defined by $f(x) = (x + I_1, x + I_2, \ldots, x + I_n)$, which is a homomorphism, show that the ideals I_i are co-prime. Moreover, prove that $\prod_{i=1}^{n} I_i = \bigcap_{i=1}^{n} I_i$ **[5 Marks]**

c) Differentiate between Noetherian and Artinian Rings. Give two examples in each case. **[5 Marks]**

d) Let K be a field and $K[x, y]$ be a polynomial ring over K. Describe the structure of any primary ideal of $K[x, y]$ **[5 Marks]**

QUESTION THREE (20 MARKS)

a) Define a Semi-Local Ring? Give 2 examples. **[4 Marks]**

b) Consider the ring $R = \mathbb{Z}$ and define a group $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. On M define addition and multiplication operations by $(a,b) + (c,d) = (a+c, b+d)$ and $k(a,b) = (ka, kb)$ where $k \in \mathbb{Z}$, show that M is a module over $\mathbb{Z}$ **[5 Marks]**

c) Let A be a ring with identity and M be a finitely generated module over A. Suppose S is a multiplicative closed subset of A, show that $S^{-1}Ann(M) = Ann(S^{-1}M)$. **[6 Marks]**

d) Let A[x] be a Noetherian Ring. Does it follow that A is Noetherian too? **[5 Marks]**

QUESTION FOUR (20 MARKS)

a) Suppose R is a principal ideal domain, show that every nonzero prime ideal of R is a maximal ideal **[5 Marks]**

b) Let M be a finitely generated $R-$module and I be an ideal of R such that $IM = M$. Show that there exists an element $x \in R$ obeying $x \equiv 1 \bmod I$ and $xM = (0)$ **[5 Marks]**

c) Let N and P be modules over a ring A with identity. Suppose P is finitely generated, show that $S^{-1}(N:P) = (S^{-1}N:S^{-1}P)$. **[5 Marks]**

d) Define a Short Exact sequence. Give two examples **[5 Marks]**

QUESTION FIVE (20 MARKS)

a) What is a radical of a ring? Let $R = R[x]$ be a power series ring, find the Nilradical and the Jacobson's radical of R. **[4 Marks]**

b) State the Isomorphism Theorems for Modules. **[3 Marks]**

c) State and prove Nakayama's Lemma **[8 Marks]**

d) Let $\phi : A \to B$ be a surjective ring homomorphism. Show that if A is Noetherian/Artinian, then B is Noetherian/Artinian. **[5 Marks]**

END OF EXAMINATION: GOOD LUCK