(The University Of Choice)

MASINDE MULIRO UNIVERSITY OF SCIENCE AND TECHNOLOGY (MMUST)

UNIVERSITY EXAMINATIONS

2021/2022 ACADEMIC YEAR

(MAIN EXAMINATION)

FOURTH YEAR SECOND SEMESTER EXAMINATION

FOR THE DEGREE OF

BACHELOR OF SCIENCE IN (MATHEMATICS)
(SMT & SME)
AND BACHELOR OF SCIENCE IN EDUCATION (ETS, EDA & EDS)

COURSE CODE: MAT 422

COURSE TITLE: PARTIAL DIFFERENTIAL EQUATION II (PDE II)

DATE: 27^{th} April, 2022 TIME: 8: 00 - 10: 00 AM

INSTRUCTIONS TO CANDIDATES:

- *Answer Question ONE (COMPULSORY) and ANY OTHER TWO questions.*
- *Do not write on the question paper.*

Time: 2 Hours

MMUST observes ZERO tolerance to examination cheating

This paper consists of 4 printed pages. Please turn over.

QUESTION ONE (COMPULSORY) **[30 MARKS]**

(a) What is a partial differential equation? **[1 mark]**

(b) Classify the following PDE as Parabolic, Hyperbolic, or Elliptic

(i) $u_{xx} + x^2 u_{yy} = 0$ **[2 marks]**

(ii) $u_{xx} - 5u_{xy} + 4u_{yy} + u_x + u_y + 5u = 2$ **[2 marks]**

(c) Determine the type of equation and transform it into its canonical form **[5 marks]**

$$u_{xx} - x^2 u_{yy} = 0$$

(d) Solve the Initial value problem using the d'Alemberts formula, **[5 marks]**

$$u_{tt} - 5^2 u_{xx} = 0 \qquad -\infty < x < \infty$$

$$u(x,0) = x^2, \qquad u_t(x,0) = cosx$$

(e) Solve the partial differential equation using the method of characteristics **[5 marks]**

$$u_y - xu_x = 1 - u \qquad u(x,0) = cosx$$

(f) Write down an explicit formula for the function u that solves the transport equation **[5 marks]**

$$u_t + b.u_x = ku \quad in \quad \mathbb{R}^n \times [0,\infty)$$

$$u = g \quad on \quad \mathbb{R}^n \times (t = 0)$$

(g) Solve wave equation by methods of separation of variables **[5 marks]**

$$\frac{\partial^2 u}{\partial t^2} = k^2 \frac{\partial^2 u}{\partial x^2} \qquad t \geq 0$$

$$u(0,t) = 0, \quad u(l,t) = 0, \qquad 0 < x < l$$

QUESTION TWO **[20 MARKS]**

(a) Using the change of variable $\xi = x + t$ and $\eta = x - t$. Show that $u_{xx} - u_{tt} = 0$ if and only if $u_{\xi\eta} = 0$ **[4 marks]**

(b) An insulated rod of length 10cm has its ends A and B maintained at 20^0 and 70^0 respectively until steady state conditions prevail. The temperature at the ends are changed to 30^0 and 50^0 respectively. Find the temperature distribution in the rod at time t **[6 marks]**

(c) Determine the regions in the xy-plane where the following equation is hyperbolic, parabolic, or elliptic **[4 marks]**

$$u_{xx} + yu_{yy} + \frac{1}{2}u_y = 0$$

(d) The temperature $T(x,t)$ in a stationary medium, $x \geq 0$, is governed by the heat conduction equation **[6 marks]**

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

Making the change of variable $(x,t) \to (u,t)$, where $u = \frac{x}{2\sqrt{t}}$ show that

$$4t\frac{\partial T}{\partial \tau} = \frac{\partial^2 T}{\partial u^2} + 2u\frac{\partial T}{\partial u}$$

QUESTION THREE **[20 MARKS]**

(a) Find the Greens function for the Dirichlet boundary value problem (BVP) **[5 marks]**

$$\nabla^2 u = u_{xx} + u_{yy} = F \quad for \quad x > 0,\, y > 0$$

(b) Solve the cauchy problem by method of characteristics **[5 marks]**

$$u_y + uu_x = 0, \qquad u(x,0) = x$$

(c) Find the eigenvalues and eigenfunctions of the Laplacian **[5 marks]**

$$\begin{aligned} u_{xx} + u_{yy} + \lambda u & \qquad in \ \Omega \\ u(0,y) = 0 = u(a,y) & \qquad 0 \leq y \leq b \\ u(x,0) = 0 = u(x,b) & \qquad 0 \leq x \leq a \end{aligned}$$

(d) Find the Fundamental solution of the Laplace equation $u_{xx} + u_{yy} = 0$ **[5 marks]**

QUESTION FOUR **[20 MARKS]**

(a) State the maximum principle for the laplace's equation **[2 marks]**

(b) Solve the non-homogeneous wave equation **[7 marks]**

$$u_{tt} - u_{xx} = xsint \qquad t > 0, \quad -\infty < x < \infty$$

$$u(x,0) = cosx, \qquad u_t(x,0) = x^2$$

(c) Reduce to canonical form **[6 marks]**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} = 0$$

(d) Suppose that the subharmonic function u satisfies

$$\nabla^2 u = F \text{ in } \Omega, \text{ with } F > 0 \text{ in } \Omega.$$

Prove that $u(x,y)$ attains its maximum on $\partial\Omega$. **[5 marks]**

QUESTION FIVE **[20 MARKS]**

(a) Suppose that $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} < 0$ in V and $u(x,0) \leq M$. **[4 marks]**
Then $u(x,t) \leq M$ in V_T

(b) Verify that $u(x,y) = ln(\sqrt{x^2+y^2})$ satisfies the equation $u_{xx} + u_{yy} = 0$ for all $(x,y) \neq (0,0)$ **[5 marks]**

(c) Solve the given PDE **[6 marks]**

$$u_t = 2^2 u_{xx} \qquad 0 < x < 2,$$

$$u(0,t) = 3 \qquad u(2,t) = 4$$

$$u(x,0) = 3, \qquad t > 0$$

(d) Using the Green's formula show that the first fundamental formula for heat equation is given by **[5 marks]**

$$\int_{\partial R} uv\, dx + (v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x})dt = 0$$