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(University of Choice)

MASINDE MULIRO UNIVERSITY OF SCIENCE AND TECHNOLOGY (MMUST)

UNIVERSITY EXAMINATIONS

2022/2023 ACADEMIC YEAR

FIRST YEAR FIRST SEMESTER EXAMINATIONS

FOR THE DEGREE
OF

MASTER OF SCIENCE (APPLIED MATHEMATICS)

COURSE CODE: MAT 807
COURSE TITLE: FUNCTIONAL ANALYSIS I

DATE: Thursday, 27th April 2023 **TIME:** 2 pm – 5 pm

INSTRUCTIONS TO CANDIDATES
Answer ANY THREE questions

Time: 3 hours

QUESTION ONE (20 MARKS)

a. Let $\{e_1, e_2, \dots, e_n\}$ be a linearly independent set of vectors in a normed linear space X of any finite dimension. Show that, in this case, there is a number $c > 0$ such that for every choice of scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ we have
$\|\alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_n e_n\| \geq c(|\alpha_1| + |\alpha_2| + \dots + |\alpha_n|), c > 0$ (15 marks)

b. Show that every finite dimensional normed linear space is complete. (5 marks)

QUESTION TWO (20 MARKS)

Consider an operator $Tx(t) = \int_a^b k(t,s)x(s)ds$, with $k(t,s)$ a continuous function of t and s $a \leq t, s \leq b$.

i. Determine whether or not the operator T is a linear.

ii. Determine whether T is bounded.

iii. Find $\|T\|$.

QUESTION THREE (20 MARKS)

a. Prove that if X is a normed linear space and Y is complete, then the space of bounded linear operators from X into Y is also complete.

b. Let X be a normed linear space into over $\mathbb{R}$ or $\mathbb{C}$. Show that if $A, B \in B(X, Y)$, then their product $AB \in B(X, Y)$ and $\|AB\| \leq \|A\|\|B\|$.

c. Let A: $\mathbb{R}^m \to \mathbb{R}^n$ where $\mathbb{R}^m$ and $\mathbb{R}^n$ have the ℓ_1 –norm, $\|x\|_1 = \sum_{i=1}^{m} |\xi_i|$ for $x = (\xi_1, \xi_2, \cdots, \xi_m)$, then $\|A\|_{\ell_1} = \max_{1 \leq j \leq n} \sum_{i=1}^{n} \|a_{ij}\|$.

QUESTION FOUR (20 MARKS)

a. Suppose X is a linear space with inner product $\langle \cdot, \cdot \rangle$. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$, prove that $\langle x_n, y_n \rangle \to \langle x, y \rangle$ as $n \to \infty$. (5 marks)

b. Let X be a normed linear space. Prove that X is a Banach space if and only if the series $\sum_{n=1}^{\infty} a_n$ converges, where (a_n) is any sequence in X satisfying $\sum_{n=1}^{\infty} \|a_n\| < \infty$. (15 marks)

QUESTION FIVE (20 MARKS)

a. Let X be an inner product space and let $\{x_1, x_2, \cdots, x_N\}$ be an orthonormal set. Prove that $\|x - \sum_{n=1}^{N} c_n x_n\|$ is minimized by choosing $c_n = \langle x_n, x \rangle$. (7 marks)

b. Show that ℓ_p, $1 \leq p < \infty$ equipped with the $\|\cdot\|_p$ is a Banach space. (8 marks)

c. State and prove Riesz Representation Theoerm. (5 marks)