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(University of Choice)

MASINDE MULIRO UNIVERSITY OF SCIENCE AND TECHNOLOGY (MMUST)

UNIVERSITY EXAMINATIONS

2022/2023 ACADEMIC YEAR

THIRD YEAR SECOND SEMESTER EXAMINATIONS

FOR THE DEGREE
OF

BACHELOR OF SCIENCE (MATHEMATICS WITH IT)

COURSE CODE: MAT 308
COURSE TITLE: RING THEORY

DATE: Friday, 14th April 2023 **TIME:** 8.00 am – 10 am

INSTRUCTIONS TO CANDIDATES

Answer question ONE (COMPULSORY) and any other TWO questions

Time: 2 hours

QUESTION ONE COMPULSORY (30 MARKS)

a. Show that in an integral domain, if $ab = ac, a \neq 0$ then $b = c$. (4 marks)

b. Let r and s be arbitrary elements of a ring R. Prove that $(-r)s = r(-s) = -(rs)$ (5 marks)

c. Find the gcd of $f = x^4 - x^2 + x - 1$ and $g = x^3 - x^2 + x - 1$ in $\mathbb{Q}[x]$ and express it as a linear combination of these polynomials. (6 marks)

d. Explain what is meant by a polynomial being irreducible. Show that $x^2 + x + 1$ is the only irreducible quadratic over $\mathbb{Z}_2$. (5 marks)

e. State the Remainder Theorem. Verify the Remainder Theorem for $f(x) = 2x^3 + x + 1 \in \mathbb{Z}_6[x]$ for $a = 2$. (5 marks)

f. Prove that the characteristic of an integral domain is a field is either 0 or a prime. (5 marks)

QUESTION TWO (20 MARKS)

a. Let $R = \mathbb{Z}(i) = \{a + bi \mid a, b \in \mathbb{Z}\}$ be the ring of Gaussian integers and let $A = (2 + i)R$ denote the ideal of all multiples of $(2 + i)$. Describe the cosets in R/A. (15 marks)

b. Prove that every finite integral domain is a field. (5 marks)

QUESTION THREE (20 MARKS)

a. Show that $m^3 - 6n^2 = 3$ has no solution in $\mathbb{Z}$. (7 marks)

b. If $\theta: \begin{bmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{R} \end{bmatrix} \rightarrow \mathbb{R} \times \mathbb{R}$ is given by $\theta \begin{bmatrix} r & s \\ 0 & t \end{bmatrix} = (r, t)$. Show that θ is an onto ring homomorphism. (7 marks)

c. Let $f = gh$ in $\mathbb{Z}[x]$. If a prime $p \in \mathbb{Z}$ divides every coefficient of f, show that either p divides every coefficient of g or p divides every coefficient of h (Gauss Lemma). (6 marks)

QUESTION FOUR (20 MARKS)

a. If R is a commutative ring, an ideal $P \neq R$ of R is a prime ideal if and only if R/P is an integral domain. (10 marks)

b. Let A be an additive subgroup fo a ring R. Prove that the following are equivalent.

   i. The multiplication $(r + A)(s + A) = rs + A$ is well defined on R/A.

   ii. $Ra \subseteq A$ and $aR \subseteq A$ for all a in A. (10 marks)

QUESTION FIVE (20 MARKS)

a. Show that $R = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$ is isomorphic to $\mathbb{C}$. (8 marks)

b. Let a, b be elements in a ring R which commute. Prove that for each $n \geq 0$,

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \ldots + \binom{n}{n-1} ab^{n-1} + \binom{n}{n} b^n$$

Where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ denotes the binomial coefficient. (Binomial Theorem). (8 marks)

c. Factor $f = x^4 - x^3 - x^2 - x - 2$ as far as possible in $\mathbb{Q}[x]$. (4 marks)