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*(The University Of Choice)*

# MASINDE MULIRO UNIVERSITY OF SCIENCE AND TECHNOLOGY (MMUST)

## UNIVERSITY EXAMINATIONS
## 2022/2023 ACADEMIC YEAR

### MAIN EXAMINATIONS FOR

### FOURTH YEAR SECOND SEMESTER EXAMINATIONS FOR THE DEGREE OF BACHELOR OF SCIENCE (SME/SMT)

COURSE CODE: MAT 402

COURSE TITLE: MEASURE THEORY

**DATE: TUESDAY 25TH APRIL, 2023 TIME: 12.00-2.00P.M**

**Instructions to candidates:**
Answer Question one and any other two questions.

Time: 2 hours

This paper consists of 3 printed pages. Please turn

## QUESTION ONE (30 MARKS)

a) Define the term Lebesgue measure. (2 Marks)

b) Show that if $E \in \mathcal{M}$ then for any given $\epsilon > 0$, there exists a closed set $F \subset E$ such that $m(E \backslash F) < \epsilon$. (4 Marks)

c) If $f$ and $g$ are measurable functions defined on $E \in \mathcal{M}$. Prove that $f + g$ is also measurable. (6 Marks)

d) If $\{f_n\}$ is a sequence of non-negative measurable functions and $\{f_n : n \geq 1\}$ increases monotonically to $f(x)$ for each $x$ that is $f_n \nearrow f$ pointwise, show that (6 Marks)

$$\lim_{n \to \infty} \int_E f_n(x)\, dm = \int_E f\, dm$$

e) Prove that if $f$ is a measurable function, then the level set $\{x : f(x) = a\}$ is measurable for every $a \in \overline{\mathbb{R}}$. (4 Marks)

f) Investigate the convergence of

$$\int_a^{\infty} \frac{n^2 x e^{-n^2 x^2}}{1 + x^2}\, dx$$

For $a > 0$, and for $a = 0$. (6 Marks)

g) State the Dominated Convergence Theorem. (2 Marks)

## QUESTION TWO (20 MARKS)

a) Suppose $f$ is a non-negative measurable function prove that $f = 0$ a.e. if and only if (7 Marks)

$$\int_{\mathbb{R}} f\, dm = 0$$

b) Show that for a sequence of non-negative measurable functions $f_n$ we have (6 Marks)

$$\int \sum_{n=1}^{\infty} f_n\, dm = \sum_{n=1}^{\infty} \int f_n\, dm.$$

c) If $f$ and $g$ are integrable, $f \leq g$, prove that $\int f\, dm \leq \int g\, dm$. (4 Marks)

d) Distinguish between an inner measure and an outer measure. (3 Marks)

## QUESTION THREE (20 MARKS)

a) Use the dominated convergence theorem to find (4 Marks)

$$\lim_{n\to\infty}\int_1^{\infty} f_n(x)dx.$$

b) Find a formula describing $m(A \cup B)$ and $m(A \cup B \cup C)$ in terms of measures of the individual sets and their intersections. (5 Marks)

c) Suppose that $A_n \in \mathcal{M}$ for all $n \geq 1$. If $A_n \subset A_{n+1}$, for all $n$, show that (11 Marks)

$$m(\bigcup_n A_n) = \lim_{n\to\infty} m(A_n)$$

**QUESTION FOUR (20 MARKS)**

a) Let $E$ be a measurable subset of $\mathbb{R}$. Show that

i) $f: E \to \mathbb{R}$ is a measurable subset if and only if both $f^+$ and $f^-$ are measurable. (5 Marks)

ii) If $f$ is measurable, then so is $|f|$ but the converse is false. (5 Marks)

b) Show that if $f$ is measurable, then the truncation of $f$

$$f^a(x) = \begin{cases} a & if\ f(x) > a \\ f(x) & if\ f(x) \leq a \end{cases}$$

is also measurable. (5 Marks)

c) Find a non-measurable function $f$ such that $f^2$ is measurable. (5 Marks)

**QUESTION FIVE (20 MARKS)**

a) Suppose $\{f_n\}$ and $f$ are non-negative and measurable. If $\{f_n\}$ increases to $f$ almost everywhere, show that

$$\int_E f_n dm \nearrow \int_E f\, dm$$

For all measurable sets $E$. (8 Marks)

b) If $\{f_n\}$ is a sequence of non-negative measurable functions, show that (12 Marks)

$$\liminf_{n\to\infty} \int_E f_n dm \geq \int_E \left(\liminf_{n\to\infty} f_n\right) dm$$