5

MASINDE MULIRO UNIVERSITY OF SCIENCE AND TECHNOLOGY (MMUST)

UNIVERSITY EXAMINATIONS
2022/2023 ACADEMIC YEAR

REGULAR EXAMINATIONS

FOURTH YEAR SECOND SEMESTER (MAIN CAMPUS)

FOR THE DEGREE

OF

BACHELOR OF SCIENCE (MATHEMATICS WITH IT)

COURSE CODE: **MAT 404**

COURSE TITLE: **TOPOLOGY II**

DATE: **Thursday 20TH April 2023**

TIME: **8.00 a.m – 10.00 a.m**

INSTRUCTIONS: **Answer questions ONE and any other TWO questions.**

QUESTION ONE (30 MARKS)

a) Define a T_2-space with respect to a topological space $(X,\Im)$ and hence show that the T_2-axiom is hereditary. (5 marks)

b) Define a normal space with respect to a topological space $(X,\Im)$ and by use of an example show that a normal space need not be a regular space. (5 marks)

c) State the second axiom of countability and hence show that every subspace of a second countable space is second countable. (3 marks)

d) Show that a topological space $(X,\Im)$ is compact if and only if every class $\{F_\alpha : \alpha \in \Lambda\}$ of closed subsets of X which satisfies the finite intersection property, has itself, a non-void intersection. (5 marks)

e) Let $(X,\Im)$ be a topological space and let σ be any subbase for $\Im$. Show that the following two statements are equivalent:

   i) $(X,\Im)$ is compact.

   ii) Every cover of X by a subfamily of σ admits a finite subcover.

(4 marks)

f) Let $\{X_\alpha : \alpha \in \Lambda\}$ be a non-void family of compact topological spaces. Then show that the Cartesian product X of these spaces is compact (in the product topology).

(4 marks)

g) If A and B are connected sets which are not separated, show that $A \cup B$ is connected.

(4 marks)

QUESTION TWO (20 MARKS)

a) Let $(X,\Im)$ be a topological space which is T_1 and E be a non-void subset of X. Show that $x \in X$ is a limit point of E if and only if every open subset G containing x contains infinitely many points of E (distinct from x).

(5 marks)

b) Prove that every metric space (X,ρ) is a normal space.

(6 marks)

c) Let $\beta_p = \{B_1, B_2, \cdots\cdots\}$ be a nested basis at p and let (a_n) be a sequence such that $a_n \in B_n \quad \forall\, n \in IN$. Show that (a_n) converges to p.

(2 marks)

d) Let $(X,\Im)$ satisfy the first axiom of countability. Show that $f : X \to Y$ is continuous at $p \in X$ if and only if it is sequentially continuous at p.

(6 marks)

e) Show that any subspace $(Y,\Im_Y)$ of a first countable space is also first countable.

(2 marks)

QUESTION THREE (20 MARKS)

a) Let $(X,\Im)$ be a topological space. Show that the following conditions are equivalent;

   i) $(X,\Im)$ is a normal space.

   ii) If H is an open subset of X containing a closed subset F of X, then there exists an open subset G of X such that $F \subseteq G \subseteq \overline{G} \subseteq H$.

(5 marks)

b) Let D be the set of dyadic fractions (fractions whose denominators are powers of 2) in the unit interval $[0,1]$ i.e. $D=\left\{\frac{1}{2},\frac{1}{4},\frac{3}{4},\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8},\ldots\ldots\right\}$. Show that D is dense in $[0,1]$ with respect to the usual topology.

(3 marks)

c) Let A and B be disjoint closed subsets of a normal topological space X. Show that there exists a continuous function $f:X\rightarrow[0,1]$ such that $f(A)=\{0\}$ and $f(B)=\{1\}$, i.e. $0\leq f(x)\leq 1$ for all $x\in[0,1]$ with $f(x)=0\;\forall\, x\in A$ and $f(x)=1\;\forall\, x\in B$.

(12 marks)

QUESTION FOUR (20MARKS)

a) Let $N(x;\varepsilon)$ be an open neighbourhood in the metric space (X,ρ) and $\rho(x,a)<\frac{\varepsilon}{3}$. Show that if $\frac{\varepsilon}{3}<\delta<\frac{2\varepsilon}{3}$, then $x\in N(a;\delta)\subset N(x;\varepsilon)$. (2 marks)

b) Let (X,ρ) be a separable metric space. Show that (X,ρ) is second countable, i.e. contains a countable base.

(4 marks)

c) Let $(X,\Im)$ be a topological space which is sequentially compact. Show that $(X,\Im)$ is also countably compact.

(4 marks)

d) Let $(X,\Im)$, $(Y,\Im^*)$ be topological spaces and E be a compact subset of X. Let $f:X\rightarrow Y$ be $\Im-\Im^*$-continuous. Show that $f(E)$ is a compact subset of Y.

(4 marks)

e) Let E be a closed subset of a compact space $(X,\Im)$. Show that E is also compact.

(3 marks)

f) Let f be a one-to-one continuous function from a compact space $(X,\Im)$ into a Hausdorff space $(Y,\Im^*)$. Show that X and $f(X)$ are homeomorphic. (3 marks)

QUESTION FIVE (20 MARKS)

a) Define the following terms with respect to a topological space $(X,\Im)$;

i) Separated sets.

ii) Connected space. (2 marks)

b) Prove that a subspace of the real line $\mathbb{R}$ is connected if and only if it is an interval. In particular $\mathbb{R}$ is connected. (6 marks)

c) Let $p\in X$ and $\sigma_p=\{A_i\}$ be the class of all connected subsets of X containing p. Let $C_p=\bigcup_i A_i$. Show that;

i) C_p is connected.

ii) If B is a connected subset of X containing p, then $B\subset C_p$.

iii) C_p is a maximal connected subset of X, i.e. a component.

(3 marks)

d) Define the term component with respect to a topological space $(X, \Im)$ and hence prove that the components of $(X, \Im)$ form a partition of X, i.e., they are disjoint and their union is X. In particular every connected subset of X is contained in some component.

(5 marks)

e) Prove that continuous images of arcwise connected sets are arcwise connected.

(4 marks)

-END-

-GOOD LUCK-