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(The University Of Choice)

MASINDE MULIRO UNIVERSITY OF SCIENCE AND TECHNOLOGY (MMUST)

(MAIN EXAMINATIONS)

UNIVERSITY EXAMINATIONS
2022/2023 ACADEMIC YEAR

FOURTH YEAR FIRST SEMESTER EXAMINATIONS

FOR THE DEGREE
OF
BACHELOR OF SCIENCE (MATHEMATICS)

COURSE CODE: MAT 408

COURSE TITLE: FIELD THEORY

DATE: **TUESDAY 20TH APRIL, 2023** **TIME: 12.00-2.00P.M**

Instructions to candidates:
Attempt Question ONE and ANY other TWO Questions.

Time: 2 hours

QUESTION ONE (30 MARKS)

a) Define a field and show that every field is an integral domain while the converse need not hold. **(5 Marks)**

b) Given an integral domain D, use localization at a saturated subset of D to construct the quotient field over D. **(5Marks)**

c) Let F be a field. Suppose F and $f(x) \in F[x]$ is an irreducible polynomial, show that there exists a field K containing F and an element α satisfying $f(\alpha) = 0$. Use an example to demonstrate the construction of a field extension of a finite field of order 2. **(7Marks)**

d) Define a Frobenius map. Show that a Frobenius map on a field F of characteristic p is an endomorphism. **(4Marks)**

e) Show that a homomorphism defined on the ring $(\mathbb{Z},+,\cdot)$ is either a zero homomorphism or an identity map. **(4Marks)**

f) State Eisenstein's criterion. Hence or otherwise, show that $x^2 + y^2 - 1 \in \mathbb{Q}[x, y]$ is irreducible. **(5Marks)**

QUESTION TWO (20 MARKS)

a) What is an algebraic element? Let E and F be fields such that E is a subfield of F. Suppose $a \in F$, show that a is algebraic over E if and only if $[E(a):E]$ is finite. **(5Marks)**

b) Let E, F, G be fields such that $E \subseteq F \subseteq G$. Prove that $[G:E]$ is finite if both $[F:E]$ and $[G:F]$ are finite and $[G:E] = [G:F][F:E]$ **(6Marks)**

c) State Gauss' Lemma. Use the Lemma to prove that $\sqrt{2}$ is irrational **(4Marks)**

d) Using an example, show that every ideal I of a ring R is a subring of R and the converse is not necessarily true. **(5Marks)**

QUESTION THREE (20 MARKS)

a) Let R be a principal Ideal Domain. If f, g are primitives in R[x], show that fg is also primitive in R[x]. **(4Marks)**

b) Consider a finite number of points say S in the Euclidean plane and some $T \subseteq S$. When is T said to be constructible in S? Demonstrate your argument. **(5Marks)**

c) Let S be a non-empty set and P(S) be the power set of S. Show that the binary operations of intersection and union are closed on P(S) so that $(P(S),\bigcup,\bigcap)$ is a magma. Moreover, suppose multiplication on P(S) is defined by $\bigcap$ and addition by $\circ: A\circ B=(A\bigcup B)\setminus(A\bigcap B)$, by setting $S=\mathbb{Z}_2$, show that $(P(\mathbb{Z}_2),\circ,\bigcap)$ forms a ring with identity. **(5Marks)**

d) Let F be an arbitrary field. Show that there exists a unique linear map $D: F(x)\to F(x)$ satisfying

i. $D(fg)=f.D(g)+D(f).g$
ii. $D(1)=0$
iii. $D(x)=1$ **(3Marks)**

e) Define: Euclidean Domain, Principal Ideal Domain, Unique Factorization Domain **(3Marks)**

QUESTION FOUR (20 MARKS)

a) Let f be an irreducible polynomial over a perfect field F of characteristic p. Show that f has no repeated roots in the extension field of F. **(7Marks)**

b) Show that a finite subgroup of a multiplicative group of a finite field is cyclic. **(7Marks)**

c) Let F be a field of characteristic p. Suppose a is an algebraic element in F, show that a is separable in F if F(a)=F(b) for some b in F. **(6Marks)**

QUESTION FIVE (20 MARKS)

a) Let q be a prime power (p^n). Show that the polynomial $x^{q^n}-x$ is a product of monic irreducible polynomials over $GF(p^n)$ whose degrees divide n **(5Marks)**

b) Prove the following statements **(5Marks)**

i. Every polynomial of odd degree over $\mathbb{R}$ has a root in $\mathbb{R}$
ii. Every positive real number has a positive square root
iii. Every complex number has a complex square root

c) State the Fundamental Theorem of Galois Theory **(4Marks)**

d) Differentiate between a normal extension and a separable extension **(3Marks)**

e) Consider the ring $\mathbb{Z}\left[\sqrt{d}\right]=\{a+b\sqrt{d}:a,b\in\mathbb{Z}\}$ with $\psi(a+b\sqrt{d})=\left|a^2-b^2d\right|$. Show that $\mathbb{Z}\left[\sqrt{d}\right]$ is a Euclidean Domain for $d=-2,-1,2,3$. **(3Marks)**

END OF EXAMINATION: GOOD LUCK