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(*University of Choice*)

MASINDE MULIRO UNIVERSITY OF SCIENCE AND TECHNOLOGY (MMUST)

UNIVERSITY EXAMINATIONS
2022/2023 ACADEMIC YEAR

FOURTH YEAR SECOND SEMESTER EXAMINATIONS
MAIN EXAM

FOR THE DEGREE OF
BACHELOR SCIENCE IN EDUCATION AND ARTS, MATHEMATICS WITH IT, MATHEMATICS WITH ECONOMICS, PHYSICS, AND STATISTICS

COURSE CODE: MAT 422

COURSE TITLE: PARTIAL DIFFERENTIAL EQUATIONS I

DATE: 19TH APRIL, 2023 **TIME:** 12.00-2.00 PM

INSTRUCTIONS TO CANDIDATES

- Answer question ONE (COMPULSORY) and any other TWO questions

This Paper Consists of ~~5~~ 4 Printed Pages. Please Turn Over.

QUESTION ONE (30 MARKS)

a) Classify the following PDEs (4 Marks)

(i) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{C^2}\frac{\partial^2 u}{\partial t^2}$

(ii) $\frac{\partial^2 u}{\partial x^2} + x^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = 0$

b) Transform the following PDE to its standard form (5 Marks)

$$y^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} + x^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{xy}\left(y^3\frac{\partial u}{\partial x} + x^3\frac{\partial u}{\partial y}\right)$$

c) State the Fundamental solution for the heat equation $u_t = u_{xx}$ (2 Marks)

d) Using the method of characteristics, solve the linear first order equation (3 Marks)

$$x^2 u_x + y u_y + xyu = 1$$

e) Using D'Alemberts method, solve the PDE

$$u_{tt} = 36u_{xx} \qquad -\infty \le x \le \infty \qquad t \ge 0$$

With initial condition

$$u(0,x) = \cos 2x \qquad u_t(0,x) = \sin x$$ (5 Marks)

f) Solve the following non-homogeneous transport problem (5 Marks)

$$u_t + b\ Du + cu = 0 \quad in\ \mathbb{R}^n \times (0,\infty)$$
$$u = g \quad on\ \mathbb{R}^n \times (t=0)$$

g) Using the change of variables $\varepsilon = x+t$ and $\eta = x-t$, show that

$$u_{tt} - u_{xx} = 0 \text{ if and only if } u_{\varepsilon\eta} = 0$$ (4 Marks)

h) Find the range of values of the solution $u(x,y) \in \bar{\Omega}$ in the following boundary value problem

$$u_{xx} + u_{yy} = 0 \quad (x,y) \in \Omega = \{(x,y): 0 < x < 2, 0 < y < 1\}$$ (4 Marks)

$$u(x,y) = \sin \pi x + \cos \pi y \quad (x,y) \in \partial\Omega$$

(i) Hence transform it into its canonical form (5 Marks)

d) Show that the Dirichlet BVP has utmost one solution (5 Marks)

QUESTION FOUR (20 MARKS)

a) State the maximum principle for Laplace equation (2 Marks)

b) Solve the following Initial Value Problem (4 Marks)

$$u_{tt} - k^2 u_{xx} = 0 \quad t > 0 \quad -\infty < x < \infty$$

$$u(0,x) = \sin x \quad u_t(0,x) = 0$$

c) Solve the following initial value problem (5 Marks)

$$u_t = 4u_{xx} \quad t \geq 0 \quad 0 \leq x \leq 2$$

$$u(t,0) = 1 \quad u(t,2) = 4 \quad t \geq 0$$

$$u(0,x) = 1 \quad 0 \leq x \leq 2$$

d) Show that the general solution of the PDE

$u_{xy} = 0$ is $u(x,y) = F(x) + G(y)$ for arbitrary functions of F and G (4 Marks)

e) Given that $u(x,t) = v\left(\frac{x}{\sqrt{t}}\right)$, show that $u_t = u_{xx}$ if and only if $v''(z) + \frac{v'(z)}{z} = 0 \quad (z > 0)$

where the prime indicates differentiation with respect to z and $z = \frac{x}{\sqrt{t}}$ (5 Marks)

QUESTION FIVE (20 MARKS)

a) A string is stretched along the x-axis to which it is attached at $x = 0$ and $x = L$. Find y in terms of x and t assuming that $y = mx(L - x)$ when $t = 0$ (5 Marks)

b) Suppose that $u(x,y)$ is a continuous function on the closed disk $r \leq 1$ and harmonic in the open disk $r < 1$. If $u(\cos\theta, \sin\theta) \leq \sin\theta + \cos 2\theta$ then show that $u(x,y) \leq y + x^2 - y^2$ for all $x^2 + y^2 \leq 1$ (5 Marks)

c) A transmission line cable 1000 miles long is initially under steady conditions with potential 1300V at sending end (*x=0*) and 1200V at the receiving end (*x=100*). The terminal end of the cable is suddenly grounded but the potential at the source is kept at 1300V. Find the potential *V(x,t)* when the inductance and leakage are negligible (5 marks)

c) Solve the following BVP (5 Marks)

$$u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = 0 \quad -\pi < \theta < \pi \quad 0 < r < 2$$

QUESTION TWO (20 MARKS)

a) Consider the equation

$$xu_{xx} - yu_{yy} + 0.5(u_x - u_y) = 0$$

Find the domain where the equation is elliptic and domain where its hyperbolic (3 Marks)

b) Determine the solution to the PDE

$$\frac{\partial v}{\partial t} + 3\frac{\partial v}{\partial x} = 0 \qquad \text{such that}$$

$$V(x,0) = \begin{cases} \frac{1}{2}x & 0 < x < 1 \\ 0 & elsewhere \end{cases}$$

And hence sketch its characteristic curve at $x = 1$ (5 Marks)

c) Using the method of separation of variables, determine the solution to the following laplace equation (6 Marks)

$$u_{xx} + u_{yy} = 0 \qquad 0 \le x \quad y \le 1$$

$$u(x,0) = u_0$$

$$u(x,1) = 0 \qquad u(0,y) = u(1,y) = 0$$

d) A string is fixed at two points L apart and is stretched. The motion takes place by displacing the string in the form $y = a\sin\left(\frac{\pi x}{L}\right)$ from which it is released at time $t = 0$.

Show that the displacement of any point at a distance x from one end at time t is

$$y(x,t) = a\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi ct}{L}\right)$$ (6 Marks)

QUESTION THREE (20 MARKS)

a) Transform the two dimensional Laplace equation $u_{xx} + u_{yy} = 0$ into its polar form (4 Marks)

b) Solve the following non-homogeneous transport problem (4 Marks)

$$u_t + bu_x = f \quad \text{in } \mathbb{R}^n \times (0,\infty)$$

$$u = g \quad \text{on } \mathbb{R}^n \times (t = 0)$$

c) (i) Find the characteristics of the following PDE (2 Marks)

$$y^2u_{xx} - x^2u_{yy} = 0 \quad x, y > 0$$

$$u(2,\theta) = \cos\frac{\theta}{4} \qquad -\pi < \theta < \pi$$

ALL THE BEST