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(The University of Choice)

MASINDE MULIRO UNIVERSITY OF SCIENCE AND TECHNOLOGY

UNIVERSITY EXAMINATIONS

2022/2023 ACADEMIC YEAR

FIRST YEAR SECOND SEMESTER MAIN EXAMINATIONS

FOR THE DEGREE

IN

MASTER OF SCIENCE (PURE MATHEMATICS)

COURSE CODE: MAT 804

COURSE TITLE: ABSTRACT INTEGRATION II

DATE: 27TH APRIL, 2023 **TIME: 8.00AM – 11.00AM**

INSTRUCTIONS TO CANDIDATES

- Section A is compulsory any other THREE questions from section B
- Do all the rough work in the answer booklet

TIME: 3 hours

QUESTION ONE (20 MARKS)

a) Show that if $\phi(E) = \int_E f \, d\mu$ where $\int f \, d\mu$ is defined then ϕ is a signed measure.

(5 Marks)

b) Define the following terms:

i) Positive set

ii) Null set (4 Marks)

c) Let ν be a signed measure on $[X, \mathcal{S}]$. Show that there exist measures ν^+ and ν^- on $[X, \mathcal{S}]$ such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$. The measures ν^+ and ν^- and uniquely defined by ν and $\nu = \nu^+ - \nu^-$ is said to be the Jordan decomposition of ν. (8 Marks)

d) Show that the signed measure ν is finite or σ –finite respectively if and only if $|\nu|$ is also, or if and only if , both ν^+ and ν^- are. (3 Marks)

QUESTION TWO (20 MARKS)

a) Let f be a non-negative $\mathcal{S} \times \mathcal{J}$-measurable function and let $\phi(x) = \int_Y f_x \, d\nu$, $\psi(y) = \int_X f^y \, d\mu$ for each $x \in X$, $y \in Y$, show that ϕ is $\mathcal{S}$ –measurable and $\int_X \phi \, d\mu = \int_{X \times Y} f \, d(\mu \times \nu) = \int_Y f^y \, d\nu$. (6 Marks)

b) Let $f(x) = \sqrt{x}$, $0 \leq x \leq \frac{1}{2}$. Let $f(1) = 0$ and define $f$to be linear on $\left[\frac{1}{2}, 1\right]$. Let $f(x + k) = f(x)$ for each $k \in \mathbb{Z}$ and each x. Show that f is continuous on $\mathbb{R}$ but not absolutely continuous. (4 Marks)

c) If E_i, $i = 1, \ldots, n$ are disjoint intervals such that $\bigcup_{i=1}^{n} E_i \subseteq I$, where I is an interval, Show that $\sum_{i=1}^{n} \mu(E_i) \leq \mu(I)$. (5 Marks)

d) Let f be a finite-valued monotone increasing function defined on (a, b). Show that

i) $g(x) = f(x^-)$ is left-continous and monotone on (a, b).

ii) $h(x) = f(x^+)$ is right-continous and monotone on (a, b).

iii) If μ is a finite measure on $[\mathbb{R}, \mathcal{B}]$, $g(x) = \mu(-\infty, x)$ and $h(x) = \mu(-\infty, x]$ then $h(x) = g(x^+)$. (6 Marks)

QUESTION THREE (20 MARKS)

a) Show that the following conditions on the signed measure μ and ν on $[X, \mathcal{S}]$ and equivalent.

i) $\nu \leq \mu$

ii) $|\nu| \leq |\mu|$

iii) $\nu^+ \leq \mu$ and $\nu^- \leq \mu$ (6 Marks)

b) If $[X, \mathcal{S}]$ is a σ −finite measure space and ν is a σ −finite measure on $\mathcal{S}$ such that $\nu \leq \mu$, then there exists a finite- valued non – negative measurable function f on X such that for each $E \in \mathcal{S}$, $\nu(E) = \int_E f\, d\mu$. Also f is unique in the sense that if $\nu(E) = \int_E g\ d\mu$ for each $E \subset \mathcal{S}$, Show that $f = g\ (\mu)$ a.e. (14 Marks)

QUESTION FOUR (20 MARKS)

a) Let f and g be absolutely continuous on the finite interval $[a, b]$. Show that fg is absolutely continuous on $[a, b]$. (4 Marks)

b) Let f be a measurable function from $[X, \mathcal{S}, \mu]$ to $\mathbb{R}$ and g a Borel measurable function on $\mathbb{R}$. Show that $\int g\ d\mu f^{-1} = \int f \circ g\ d\mu$, in the sense that if either exists, so does the other and the tw are equal. (5 Marks)

c) Show that if $f \in L(-\infty, \infty)$, then g defined by $g(x) = f(-x)$ is an integrable function and $\int_a^b f\ dx = \int_{-b}^{-a} g\ dx$ for $-\infty \leq a < b \leq \infty$. (4 Marks)

d) What is a positive linear fuctional? (2 Marks)

e) Show that if $f, g \in C(I)$, $f \geq g$ and G is a positive linear functional on $C(I)$, then $G(f) \geq G(g)$. (5 Marks)

QUESTION FIVE (20 MARKS)

a) Show that the condition $f \in L(\mu \times \nu)$ in Fubini's theorem is necessary if the order of integration is to be interchangeable. (7 Marks)

b) Let μ be a σ −finite measure and ν a σ −finite signed measure and let $\nu \ll \mu$, show that $d|\nu|/d\mu = \left|\frac{d\nu}{d\mu}\right|\ [\mu]$. (4 Marks)

c) Prove that a countable union of sets positive with respect to a signed measure ν is a positive set. (6 Marks)

d) State the Lebesgue decomposition theorem. (3 Marks)