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(The University Of Choice)

MASINDE MULIRO UNIVERSITY OF SCIENCE AND TECHNOLOGY (MMUST)

UNIVERSITY EXAMINATIONS

2022/2023 ACADEMIC YEAR

(MAIN EXAMINATION)

FOURTH YEAR SECOND SEMESTER EXAMINATION

FOR THE DEGREE OF

BACHELOR OF SCIENCE IN MATHEMATICS WITH IT (SMT)

COURSE CODE: MAT 432

COURSE TITLE: BIFURCATION AND DYNAMICS

DATE: 24^{th} **April, 2023** TIME: **8 : 00 - 10 : 00 AM**

INSTRUCTIONS TO CANDIDATES:

- *Answer Question ONE (COMPULSORY) and ANY OTHER TWO questions.*
- *Do not write on the question paper.*

Time: 2 Hours

MMUST observes ZERO tolerance to examination cheating

This paper consists of 3 printed pages. Please turn over.

QUESTION ONE (COMPULSORY) **[30 MARKS]**

(a) What is Bifurcation as used in this context? **[2 marks]**

(b) Differentiate between a hyperbolic fixed point and non-hyperbolic fixed point **[3 marks]**

(c) Derive the necessary and sufficient conditions under which a one parameter family of one dimensional vector field below to undergo a saddle-node bifurcation at a fixed point $(\mu, x) = (0,0)$ $\dot{x} = f(\mu, x)$, $x \in \mathbb{R}^1, \mu \in \mathbb{R}^1$ **[5 marks]**

(d) Consider the logistic equation describing a certain fish population $\dot{P} = 2P(1-P) - \frac{E}{2}$

(i) Find the fixed points for positive values of E. **[2 marks]**

(ii) Discuss the behaviour of the equilibrium solutions as E is varied. **[3 marks]**

(e) Analyze the following logistic system $\dot{N} = rN - r\frac{N^2}{K}$, does $\lim_{t\to\infty} N(t) = K$? **[5 marks]**

(f) Classify all the critical points of the two-dimensional (decoupled) system **[5 marks]**

$$\begin{aligned} x_1' &= -x_1 + x_1^3 \\ x_2' &= -2x_2 \end{aligned}$$

(g) Consider $\dot{x} = f(r, x) = x^2 - rx$, where r is a parameter, discuss the stability of a fixed points of $f(r, x)$, and sketch the bifurcation diagram on the $r - x$ plane. **[5 marks]**

QUESTION TWO **[20 MARKS]**

(a) Solve the following equations analytically, draw the bifurcation diagrams and identify the type of bifurcation

(i) $\dot{y} = a - y^2$ **[5 marks]**

(ii) $\dot{y} = ay - y^3$ **[6 marks]**

(b) Use polar coordinate to analyze the system below and show that there exists an attracting limit cycle **[9 marks]**

$$\begin{aligned} \dot{x} &= x - y - x(x^2 + y^2) \\ \dot{y} &= x + y - y(x^2 + y^2) \end{aligned}$$

QUESTION THREE **[20 MARKS]**

(a) Define a limit cycle. **[2 marks]**

(b) Consider the vector field below, compute its center manifold and state whether is stable or unstable **[6 marks]**

$$\begin{aligned} \dot{x} &= -xy - x^6 \\ \dot{y} &= x^2 - y \end{aligned}$$

(c) Consider the logistic model below

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}), \qquad N(0) = N_0$$

(i) Find the equilibrium points and determine their stability **[5 marks]**

(ii) Obtain the exact solution of the logistic model above. **[5 marks]**

(iii) Graph the solution of the model **[2 marks]**

QUESTION FOUR **[20 MARKS]**

(a) State the Poincare-Bendixson theorem **[2 marks]**

(b) Show that $\gamma(t) = [\cos(t), \sin(t)]^T$ represents a cycle Γ of the system **[8 marks]**

$$\begin{aligned} \dot{x} &= -y + x(1 - x^2 - y^2) \\ \dot{y} &= x + y(1 - x^2 - y^2) \end{aligned}$$

Determine the Poincare map $P(r_0)$ and show that $\dot{P}(1) = e^{(-4\pi)}$

(c) Consider the system

$$\begin{aligned} \dot{x} &= 2y \\ \dot{y} &= 12x - 3x^2 \end{aligned}$$

(i) Find the Hamiltonian $H(x, y)$ of the system above. **[4 marks]**

(ii) Find all the fixed points and sketch the phase portrait. **[6 marks]**

QUESTION FIVE **[20 MARKS]**

(a) Let x and y be the solutions of two species at time t. Assume that the interaction of the species is governed by the system of equations

$$\begin{aligned} \dot{x} &= x(-20 - x + 2y) \\ \dot{y} &= y(-50 + x - y) \end{aligned}$$

(i) Find all the critical points of the system above. **[4 marks]**

(ii) Determine the type of stability of each of the critical points in (i) above **[6 marks]**

(iii) Sketch the phase plane and clearly indicate the direction of the vector field defined by the system. **[3 marks]**

(b) Sketch the limit cycles in the following systems and determine their stability

(i) $\dot{r} = r(r-1)(r-2), \quad \dot{\theta} = 1$ **[4 marks]**

(ii) $\dot{r} = r^3 - 2r^2 + r, \quad \dot{\theta} = 1$ **[3 marks]**