(The University Of Choice)

MASINDE MULIRO UNIVERSITY OF SCIENCE AND TECHNOLOGY (MMUST)

UNIVERSITY EXAMINATIONS

2022/2023 ACADEMIC YEAR

MAIN EXAMINATIONS

FIRST SEMESTER EXAMINATION FOR MASTER OF SCIENCE IN APPLIED MATHEMATICS

COURSE CODE: MAT 863
COURSE TITLE: NUMERICAL ANALYSIS II

TIME: 3 HOURS DATE: 20TH APRIL 2023, 8:00AM - 11:00AM

Instruction to the candidates:
Answer question ONE (COMPULSORY) and any other TWO questions
Time: 3 hours

This paper consists of 4 printed pages. Please turn over.

SECTION I: Answer ALL the questions in this section

QUESTION ONE - 30 MARKS (COMPULSORY)

(a) Compute the coefficient matrix and the right hand side of the $N-$parameter Ritz approximation of the equation

$$-\frac{d}{dx}\left[(1+x)\frac{du}{dx}\right] = 0 \qquad \text{for} \quad 0 < x < 1$$

$$u(0) = 0, \quad u(1) = 1$$

Use algebraic polynomials for approximation functions. Specialize your result for $N = 2$ and compute the Ritz coefficients. **[5 mks]**

(b) Construct the weak form of the nonlinear equation

$$-\frac{d}{dx}\left(1 + 2x^2\frac{du}{dx}\right) + u = x^2 \text{ for } 0 < x < 1$$

$$u(0) = 1 \quad \left(\frac{du}{dx}\right)\bigg|_{x=1} = 2$$

. **[5 mks]**

(c) Consider the following differential equation

$$-\frac{d^2u}{dx^2} - u + x^2 = 0, \quad u(0) = 0, \; u'(1) = 1$$

Solve the above equation using

(i) The Galerkin method and **[3 mks]**

(ii) The collocation method. **[2 mks]**

(d) Give a one-parameter Galerkin solution of the equation

$$-\nabla^2 u = 1 \text{ in } \Omega \text{ (=unit square)}$$

$$u = 0 \text{ on } \Gamma$$

using trigonometric approximation functions. **[5 mks]**

(e) Solve the Poisson equation governing heat conduction in a square region:

$$-k\nabla^2 T = g_0$$

$$T = 0 \text{ on sides } x = 1 \text{ and } y = 1$$

$$\frac{\partial T}{\partial n} = 0 \text{ (insulated) on sides } x = 0 \text{ and } y = 0$$

using a one-parameter Ritz approximation of the form **[5 mks]**

$$T_1(x, y) = c_1(1 - x^2)(1 - y^2).$$

(f) Construct the variational form and the finite element model of the differential equation

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) - b\frac{du}{dx} = f \text{ for } 0 < x < L$$

over a typical element $\Omega_e = (x_a, x_b)$. Here a, b and f are known functions of x, and u is the independent variable. **[5 mks]**

SECTION II: Answer any TWO questions from this section

QUESTION TWO - 15 MARKS

(a) Solve the equation

$$\frac{d^2u}{dx^2} = 2x \text{ for } 0 < x < 1$$

subject to the boundary conditions

$$u(0) = 1 \text{ and } u(1) = 0$$

considering only the one-parameter approximation using

(i) the Galerkin method, [5 mks]

(ii) the least-squares method and [5 mks]

(iii) the Petrov-Galerkin method with weight function $w = 1$ [5 mks]

QUESTION THREE - 15 MARKS

(a) Solve the problem described by the following equation

$$-\frac{d^2u}{dx^2} = \cos(\pi x), \quad 0 < x < 1; \quad u(0) = 0, \; u(1) = 0.$$

Use the uniform mesh of three linear elements to solve the problem and compare it against the exact solution [10 mks]

$$u(x) = \frac{1}{\pi^2}(cos\pi x + 2x + 1).$$

(b) Consider a uniform bar of cross-sectional area A, modulus of elasticity E, mass density m, and length L. The axial displacement under the action of time-dependent axial forces is governed by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}, \quad a = \left(\frac{E}{m}\right)^{1/2}.$$

Determine the transient response [i.e., find $u(x,\ t)$] of the bar when the end $x = 0$ is fixed and the end $x = L$ is subjected to a force P_0. Assume zero initial conditions. Use one linear element to approximate the spatial variation of the solution, and solve the resulting ordinary differential equation in time exactly to obtain [5 mks]

$$u_2(x,\ t) = \frac{P_oL}{AE}\frac{x}{L}(1 - cos\alpha t), \quad \alpha = \sqrt{3}\frac{3}{L}.$$

QUESTION FOUR - 15 MARKS

The transient heat conduction problem is described by the differential equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for } 0 < x < 1$$

and boundary conditions

$$u(0,\ t) = 0, \quad \frac{\partial u}{\partial x}(1,\ t) = 0$$

and initial condition

$$u(x,\ 0) = 1.0$$

where u is the nondimensionalized temperature. Find the finite element solution to this problem using linear interpolating functions. [15 mks]

QUESTION FIVE - 15 MARKS

Consider the following pair of differential equations:

$$-\frac{d}{dx}\left(a\frac{du}{dx} - b\frac{d^2w}{dx^2}\right) = 0, \quad -\frac{d^2}{dx^2}\left(b\frac{du}{dx} - c\frac{d^2w}{dx^2}\right) - f = 0$$

where u and w are the dependent unknowns, a, b, c and f are given functions of x.

(a) Develop the weak forms of the equations over a typical element and identify the primary and secondary variables of the formulation. Make sure that the bilinear form is symmetric (so that the element coefficient matrix is symmetric). **[5 mks]**

(b) Develop the finite element model by assuming approximation of the form

$$u(x) = \sum_{j=1}^{m} u_j\psi_j(x), \quad w(x) = \sum_{j=1}^{n} w_j\phi_j(x)$$

Hint: The weight functions v_1 and v_2 used for the two equations are like u and w, respectively. **[10 mks]**